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New recurrent algorithm for a matrix inversion

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Abstract

This paper proposes a simple approach for the matrix inversion. The number of steps is equal to the order of the matrix to be inverted. The algorithm was obtained as a "side product" during an analysis of the properties of the Guyan reduction (AIAA J. 3(2) (1965) 380). Numerical tests show an acceptable accuracy of the approach. © 2001 Elsevier Science B.V. All rights reserved.

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1. Description of the algorithm

Consider the matrix in the form

$$\boldsymbol{B}_0 = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{I} \\ \boldsymbol{I} & -\boldsymbol{A} \end{bmatrix}, \tag{1}$$

where $\boldsymbol{0}$ is zero matrix, $\boldsymbol{I}$ is unit matrix and $\boldsymbol{A}$ is $(n \times n)$ arbitrary regular matrix. The recurrent algorithm for the inversion of the matrix $\boldsymbol{A}$ can be formally expressed as follows:

For $n-1 \geqslant k \geqslant 0$,

$$\boldsymbol{B}_{k+1} = \boldsymbol{T}_{\mathrm{L}k} \boldsymbol{B}_k \boldsymbol{T}_{\mathrm{R}k}. \tag{2}$$

This algorithm eliminates one row and one column from the matrix $\boldsymbol{B}_k$ in each step. After n steps

$$\boldsymbol{B}_n = \boldsymbol{A}^{-1}. \tag{3}$$

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Depending on the pivot position, characterised by the pth row and the qth column of the matrix $\boldsymbol{B}_k$, the next possibilities are considered:

(a) $p = q = 2n - k$ (the simplest case)

$$\boldsymbol{B}_{k+1} = [\,\boldsymbol{I} \;\; -\boldsymbol{b}_{12}b_{22}^{-1}\,] \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{b}_{12} \\ \boldsymbol{b}_{21} & b_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ -b_{22}^{-1}\boldsymbol{b}_{21} \end{bmatrix} = \boldsymbol{B}_{11} - \boldsymbol{b}_{12}b_{22}^{-1}\boldsymbol{b}_{21}, \tag{4}$$

where $\boldsymbol{b}_{12}$, $\boldsymbol{b}_{21}$ are column, row vectors, respectively.

(b) $2n - k > p > n$, $q = 2n - k$

$$\boldsymbol{B}_{k+1} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{b}_{12}b_{22}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{b}_{32}b_{22}^{-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{b}_{12} \\ \boldsymbol{b}_{21} & b_{22} \\ \boldsymbol{B}_{31} & \boldsymbol{b}_{32} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ -b_{22}^{-1}\boldsymbol{b}_{21} \end{bmatrix} = \begin{bmatrix} \boldsymbol{B}_{11} \\ \boldsymbol{B}_{31} \end{bmatrix} - \begin{Bmatrix} \boldsymbol{b}_{12} \\ \boldsymbol{b}_{32} \end{Bmatrix} (b_{22}^{-1})\boldsymbol{b}_{21}. \tag{5}$$

(c) $p = 2n - k$, $2n - k > q > n$

$$\boldsymbol{B}_{k+1} = [\,\boldsymbol{I} \quad -\boldsymbol{b}_{12}b_{22}^{-1}\,] \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{b}_{12} & \boldsymbol{B}_{13} \\ \boldsymbol{b}_{21} & b_{22} & \boldsymbol{b}_{23} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -b_{22}^{-1}\boldsymbol{b}_{21} & -b_{22}^{-1}\boldsymbol{b}_{23} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{6}$$

or

$$\boldsymbol{B}_{k+1} = [\,\boldsymbol{B}_{11} \quad \boldsymbol{B}_{13}\,] - \boldsymbol{b}_{12}(b_{22}^{-1})\{\,\boldsymbol{b}_{21} \quad \boldsymbol{b}_{23}\,\}. \tag{7}$$

(d) $2n - k > p > n$, $2n - k > q > n$

Now, the matrix $\boldsymbol{B}_{k+1}$ is formally expressed by

$$\boldsymbol{B}_{k+1} = \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{b}_{12}b_{22}^{-1} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{b}_{32}b_{22}^{-1} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{b}_{12} & \boldsymbol{B}_{13} \\ \boldsymbol{b}_{21} & b_{22} & \boldsymbol{b}_{23} \\ \boldsymbol{B}_{31} & \boldsymbol{b}_{32} & \boldsymbol{B}_{33} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & \boldsymbol{0} \\ -b_{22}^{-1}\boldsymbol{b}_{21} & -b_{22}^{-1}\boldsymbol{b}_{23} \\ \boldsymbol{0} & \boldsymbol{I} \end{bmatrix} \tag{8}$$

or really

$$\boldsymbol{B}_{k+1} = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{13} \\ \boldsymbol{B}_{31} & \boldsymbol{B}_{33} \end{bmatrix} - \begin{Bmatrix} \boldsymbol{b}_{12} \\ \boldsymbol{b}_{32} \end{Bmatrix} \cdot (b_{22}^{-1}) \cdot \{\,\boldsymbol{b}_{21} \quad \boldsymbol{b}_{23}\,\}. \tag{9}$$

2. Verification

Let us consider the simple case. Now the matrix $\boldsymbol{B}_0$ is partitioned into four square submatrices

$$\boldsymbol{B}_0 = \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix} \tag{10}$$

and after transformation in one step

$$\widetilde{\boldsymbol{B}} = [\,\boldsymbol{I} \quad -\boldsymbol{B}_{12}\boldsymbol{B}_{22}^{-1}\,] \begin{bmatrix} \boldsymbol{B}_{11} & \boldsymbol{B}_{12} \\ \boldsymbol{B}_{21} & \boldsymbol{B}_{22} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{21} \end{bmatrix} = \boldsymbol{B}_{11} - \boldsymbol{B}_{12}\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{21}. \tag{11}$$

Taking into account that

$$\boldsymbol{B}_{11} = \boldsymbol{0}, \quad \boldsymbol{B}_{12} = \boldsymbol{B}_{21} = \boldsymbol{I} \quad \text{and} \quad \boldsymbol{B}_{22} = -\boldsymbol{A},$$

$$\tilde{\boldsymbol{B}} = \boldsymbol{B}_n = \boldsymbol{A}^{-1}. \tag{12}$$

For the verification of the presented algorithm it is enough to show that it gives the same results in two different cases, in one and also in two elimination steps for example. Now consider a new partitioning into the submatrices

$$\boldsymbol{B}_0 = \boldsymbol{D} = \begin{bmatrix} \boldsymbol{D}_{11} & \boldsymbol{D}_{12} & \boldsymbol{D}_{13} \\ \boldsymbol{D}_{21} & \boldsymbol{D}_{22} & \boldsymbol{D}_{23} \\ \boldsymbol{D}_{31} & \boldsymbol{D}_{32} & \boldsymbol{D}_{33} \end{bmatrix} \tag{13}$$

with the simplified presumptions, that $\boldsymbol{D}_{22}$ and $\boldsymbol{D}_{33}$ are regular and the orders of the submatrices $\boldsymbol{D}_{11}$, $\boldsymbol{D}_{22}$ and $\boldsymbol{D}_{33}$ are n_1, n_2 and n_3. Note that $n_1 = n = n_2 + n_3$.

Transformation in the first step can be expressed by

$$\boldsymbol{B}_{\mathrm{F}} = \boldsymbol{T}_{\mathrm{LF}} \boldsymbol{B}_0 \boldsymbol{T}_{\mathrm{RF}} \tag{14}$$

with the transformation matrices in the form

$$\boldsymbol{T}_{\mathrm{LF}} = \begin{bmatrix} \boldsymbol{I}_{n_1+n_2} & \begin{matrix} -\boldsymbol{D}_{13}\boldsymbol{D}_{33}^{-1} \\ -\boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1} \end{matrix} \end{bmatrix} \tag{15}$$

and

$$\boldsymbol{T}_{\mathrm{RF}} = \begin{bmatrix} \boldsymbol{I}_{n_1+n_2} \\ -\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{31}, \; -\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32} \end{bmatrix}, \tag{16}$$

where $\boldsymbol{I}_{n_1+n_2}$ is the unit matrix of order $(n_1 + n_2)$. Then

$$\boldsymbol{B}_{\mathrm{F}} = \begin{bmatrix} \boldsymbol{D}_{11} - \boldsymbol{D}_{13}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{31} & \boldsymbol{D}_{12} - \boldsymbol{D}_{13}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32} \\ \boldsymbol{D}_{21} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{31} & \boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32} \end{bmatrix}. \tag{17}$$

The second step is

$$\boldsymbol{B}_{\mathrm{S}} = \boldsymbol{T}_{\mathrm{LS}} \boldsymbol{B}_{\mathrm{F}} \boldsymbol{T}_{\mathrm{RS}} \tag{18}$$

and the transformation matrices take the form

$$\boldsymbol{T}_{\mathrm{LS}} = [\, \boldsymbol{I}_{n_1}, -(\boldsymbol{D}_{12} - \boldsymbol{D}_{13}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1} \,], \tag{19}$$

$$\boldsymbol{T}_{\mathrm{RS}} = \begin{bmatrix} \boldsymbol{I}_{n_1} \\ -(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1}(\boldsymbol{D}_{21} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{31}) \end{bmatrix}, \tag{20}$$

if the matrix $(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})$ is also regular. Final transformation matrices are

$$\boldsymbol{T}_{\mathrm{L}} = \boldsymbol{T}_{\mathrm{LS}} \boldsymbol{T}_{\mathrm{LF}} \tag{21}$$

and

$$\boldsymbol{T}_{\mathrm{R}} = \boldsymbol{T}_{\mathrm{RF}} \boldsymbol{T}_{\mathrm{RS}} = \begin{bmatrix} \boldsymbol{I}_{n_1} \\ -(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1}(\boldsymbol{D}_{21} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{31}) \\ -\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{31} + \boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32}(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1}(\boldsymbol{D}_{21} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{31}) \end{bmatrix}. \tag{22}$$

Now, the above transformation will be written in one step. Taking into account the block-matrix (10)

$$\boldsymbol{B}_{22} = \begin{bmatrix} \boldsymbol{D}_{22} & \boldsymbol{D}_{23} \\ \boldsymbol{D}_{32} & \boldsymbol{D}_{33} \end{bmatrix}, \tag{23}$$

it holds that

$$\boldsymbol{B}_{22}^{-1} = \begin{bmatrix} (\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1} & -\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23}(\boldsymbol{D}_{33} - \boldsymbol{D}_{32}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23})^{-1} \\ -\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32}(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1} & (\boldsymbol{D}_{33} - \boldsymbol{D}_{32}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23})^{-1} \end{bmatrix} \tag{24}$$

if all inverted submatrices are regular. The elimination in one step can be formally expressed by a transformation

$$\boldsymbol{B}_n = \boldsymbol{T}_{\mathrm{L}}\boldsymbol{D}\boldsymbol{T}_{\mathrm{R}} = \boldsymbol{A}^{-1} \tag{25}$$

with

$$\boldsymbol{T}_{\mathrm{L}} = [\,\boldsymbol{I}_n, \ -[\,\boldsymbol{D}_{12} \quad \boldsymbol{D}_{13}\,]\boldsymbol{B}_{22}^{-1}\,], \quad \boldsymbol{T}_{\mathrm{R}} = \begin{bmatrix} \boldsymbol{I}_n \\ -\boldsymbol{B}_{22}^{-1}\begin{bmatrix} \boldsymbol{D}_{21} \\ \boldsymbol{D}_{31} \end{bmatrix} \end{bmatrix} \tag{26}$$

or more in detail

$$\boldsymbol{T}_{\mathrm{R}} = \begin{bmatrix} \boldsymbol{I}_n \\ -(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1}\boldsymbol{D}_{21} + \boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23}(\boldsymbol{D}_{33} - \boldsymbol{D}_{32}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23})^{-1}\boldsymbol{D}_{31} \\ +\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32}(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1}\boldsymbol{D}_{21} - (\boldsymbol{D}_{33} - \boldsymbol{D}_{32}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23})^{-1}\boldsymbol{D}_{31} \end{bmatrix}. \tag{27}$$

Considering the equation

$$\boldsymbol{B}_{22}^{-1}\boldsymbol{B}_{22} = \boldsymbol{I}_n \tag{28}$$

we have the equations

$$\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23}(\boldsymbol{D}_{33} - \boldsymbol{D}_{32}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23})^{-1} = (\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1}\boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1} \tag{29}$$

and

$$(\boldsymbol{D}_{33} - \boldsymbol{D}_{32}\boldsymbol{D}_{22}^{-1}\boldsymbol{D}_{23})^{-1} = \boldsymbol{D}_{33}^{-1} + \boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32}(\boldsymbol{D}_{22} - \boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}\boldsymbol{D}_{32})^{-1}\boldsymbol{D}_{23}\boldsymbol{D}_{33}^{-1}. \tag{30}$$

After substituting relations (29) and (30) into Eq. (27), one obtains the same right transformation matrix as in Eq. (22). A similar approach can be used for the left transformation matrix. Other, more complicated cases can be verified analogously.

3. Example

As the numerical example we consider the matrix

$$\boldsymbol{A} = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 1 & -1 & 0 & 0 \\ 9 & -14 & -16 & 4 \\ -8 & 12 & 12 & -3 \end{bmatrix}$$

and, consequently,

$$B_0 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & -2 & -1 & 0 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -9 & 14 & 16 & -4 \\ 0 & 0 & 0 & 1 & 8 & -12 & -12 & 3 \end{bmatrix}.$$

For the chosen pivot b_{22} in $p=7$ and $q=8$ position Eq. (5) can be used and

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 8 & -12 & -12 \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 3 \end{Bmatrix} (-4)^{-1} \{0 \quad 0 \quad 1 \quad 0 \quad -9 \quad 14 \quad 16\},$$

i.e.,

$$B_1 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.25 & 0 & -2.25 & 3.5 & 4 \\ 1 & 0 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0.75 & 1 & 1.25 & -1.5 & 0 \end{bmatrix}.$$

Considering a new pivot position $p=6$, $q=5$ Eq. (9) gives

$$B_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0.25 & 0 & 3.5 & 4 \\ 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 0 & 0.75 & 1 & -1.5 & 0 \end{bmatrix} - \begin{Bmatrix} 1 \\ 0 \\ 0 \\ -2.25 \\ 0 \\ 1.25 \end{Bmatrix} (-1)^{-1} \{0 \quad 1 \quad 0 \quad 0 \quad 1 \quad 0\}$$

and finally

$$B_2 = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2.25 & 0.25 & 0 & 1.25 & 4 \\ 1 & 0 & 0 & 0 & -2 & -1 \\ 0 & 1.25 & 0.75 & 1 & -0.25 & 0 \end{bmatrix}.$$

In the next step, the pivot position will be $p=6$, $q=5$ and with the help of Eq. (6)

$$B_3 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & -2.25 & 0.25 & 0 & 4 \\ 1 & 0 & 0 & 0 & -1 \end{bmatrix} - \begin{Bmatrix} 1 \\ 1 \\ 0 \\ 1.25 \\ -2 \end{Bmatrix} (-0.25)^{-1} \{ 0 \quad 1.25 \quad 0.75 \quad 1 \quad 0 \},$$

i.e.,

$$B_3 = \begin{bmatrix} 0 & 6 & 3 & 4 & 0 \\ 0 & 5 & 3 & 4 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 4 & 4 & 5 & 4 \\ 1 & -10 & -6 & -8 & -1 \end{bmatrix}.$$

In the last step the 5th row and the 5th column must be eliminated using Eq. (4) and

$$B_4 = \begin{bmatrix} 0 & 6 & 3 & 4 \\ 0 & 5 & 3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 4 & 5 \end{bmatrix} - \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 4 \end{Bmatrix} (-1)^{-1} \{ 1 \ -10 \ -6 \ -8 \} = \begin{bmatrix} 0 & 6 & 3 & 4 \\ 0 & 5 & 3 & 4 \\ 1 & -10 & -6 & -8 \\ 4 & -36 & -20 & -27 \end{bmatrix} = A^{-1}$$

is obtained.

4. Numerical experiments

Following the book [1], two types of tests for a comparison of the described algorithm with the inversion via the familiar Crout (or Gauss–Doolitle) LU decomposition, were chosen:

(a) *The comparison for badly conditioned real matrices*

The square Hilbert matrix with elements $H(i,j)=(i+j-1)^{-1}$ is widely used for testing inversion algorithms and linear equation solvers. Not only is it symmetric positive definite, but it is also totally positive. It is very ill-conditioned for even moderate values of order n (see Table 1).

Table 1
Relative errors of the Hilbert matrices inversion

Order n	Condition number $cond(\boldsymbol{A})$	Crout algorithm (with row pivoting) ε_C	Mazúch–Kozánek algorithm (with row pivoting) ε_{MK}
1	1.0000e+000	0.	0.
2	1.9281e+001	0.	0.
3	5.2406e+002	0.00000000000001	0.00000000000001
4	1.5514e+004	0.00000000000022	0.00000000000041
5	4.7661e+005	0.00000000000901	0.00000000000722
6	1.4951e+007	0.00000000041513	0.00000000026564
7	4.7537e+008	0.00000001911584	0.00000000401315
8	1.5258e+010	0.00000054328636	0.00000036486362
9	4.9315e+011	0.00008646233387	0.00002565601647
10	1.6025e+013	0.00190600987246	0.00082524321093

Table 2
Average relative errors of inversion of complex matrices

Order n	Crout algorithm (with row pivoting) $m_a(\varepsilon_C) \times 10^{13}$	Mazúch–Kozánek algorithm (with row pivoting) $m_a(\varepsilon_{MK}) \times 10^{13}$
10	0.0064	0.0051
20	0.0127	0.0103
30	0.0194	0.0165
40	0.0277	0.0242
50	0.0315	0.0259

The condition number is defined by

$$cond(\boldsymbol{A}) = \|\boldsymbol{A}\| \cdot \|\boldsymbol{A}^{-1}\|,$$

the relative error of matrix inversion

$$\varepsilon = \|\boldsymbol{A}^{-1} \cdot \boldsymbol{A} - \boldsymbol{I}\| / \|\boldsymbol{A}\|$$

and the spectral norms are used in the table.

(b) *The comparison for complex matrices*

For better illustration of capabilities of the presented algorithm, the comparison of inversion accuracy for 250 square complex matrices is presented. For each given order n, 50 matrices have been randomly generated. Average relative errors of inversions (calculated as arithmetic mean m_a) are given in Table 2.

5. Concluding remarks

Comparisons presented in the above tables (resulting from inversion of 260 matrices), show acceptable stability of the recurrent algorithm. At practical realisations of the proposed algorithm on computers, it is not necessary to work explicitly with zero elements resulting from the submatrices **0** and $\boldsymbol{I}$ in the matrix $\boldsymbol{B}_k$. Therefore the next advantage of the algorithm lies in its efficiency, because it does not require more operations than the inversion via the LU decomposition. A disadvantage of the presented method lies in the fact that it requires a big programming effort for its effective realisation on computers.

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